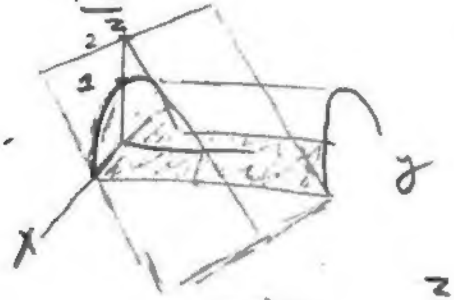
Recall Divergence Thm

If R & ∂R are nice and $\vec{F}$ is a vf on $\mathbb{R}^3$ with cts partial derivatives; then

$$\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV.$$

<u>Ex.</u> Compute the Flux of $\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$ across S the surface of region $z = 1-x^2$, $z=0$, $y=0$, $y+z=2$

Pic



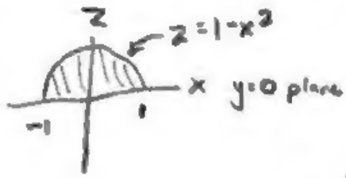
<u>Sol:</u> $\operatorname{div}(\vec{F}) = \frac{d}{dx}(xy) + \frac{d}{dy}(y^2 + e^{xz^2}) + \frac{d}{dz}(\sin(xy))$

$= y + 2y + 0 = 3y$

$y = 2-z$

$R = \{(x,y,z) : -1 \le x \le 1,\ 0 \le z \le 1-x^2,\ 0 \le y \le 2-z$

region



$$\iiint_R \operatorname{div}(\vec{F})\, dV$$

$$= 3\int_{x=-1}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{2-z} y\, dy\, dz\, dx$$

$$= \frac{3}{2}\int_{-1}^{1}\int_{0}^{1-x^2} y^2 \Big|_0^{2-z} dz\, dx$$

$$= \frac{3}{2}\int_{-1}^{1} -\frac{1}{3}(2-z)^3 \Big|_0^{1-x^2} dx$$

$$= \frac{3}{2}\cdot -\frac{1}{3}\int_{-1}^{1} (1+x^2)^3 - 8\, dx$$

$$= -\frac{1}{2}\int_{-1}^{1} (1 + 3x^2 + 3x^4 + x^6 - 8)\, dx$$

odd function $f(-x) = f(x)$

$$= -\frac{1}{2}\left(-7x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7\right)\Big|_{-1}^{1}$$

$$= -\frac{1}{2}\cdot 2\left(-7 + 1 + \frac{3}{5} + \frac{1}{7}\right) \blacksquare$$

Ex. Compute Flux of $\vec{F} = \langle xye^z, xy^2z^3, -ye^z \rangle$ across the box bounded by coordinate planes & $x=3 \quad y=2 \quad z=1$

sol: $R = [0,3] \times [0,2] \times [0,1]$

$$\text{div}(\vec{F}) = \frac{d}{dx}(xye^z) + \frac{d}{dy}(xy^2z^3) + \frac{d}{dz}(-ye^z)$$

$$= ye^z + 2xyz^3 - ye^z$$

$$= 2xyz^3$$

$$F = 2\int_0^3 \int_0^2 \int_0^1 xyz^3 \, dz \, dy \, dx$$

$$= \cancel{2} \cdot \frac{1}{\cancel{4}_2} \cdot 2 \cdot \frac{9}{\cancel{2}} = \frac{9}{2} \quad \blacksquare$$

Ex. Compute Flux of $\vec{F} = \langle 2x^3 + y^3, y^3 + z^3, 3y^2 z \rangle$ across surface of region bounded by the paraboloid $z = 1 - x^2 - y^2$ and plane $z = -3$

Pic

Shadow $z=-3$

$x^2+y^2=4$

Sol. $\text{div}(\vec{F}) = 6x^2 + 3y^2 + 3y^2$

$= 6(x^2+y^2) = 6r$ $\qquad z = 1 - x^2 - y^2$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$R(r,\theta,z)$: $0 < r < 2$, $0 < \theta < 2\pi$, $-3 < z < 1 - r^2$

$$\int_0^{2\pi} \int_0^2 \int_{-3}^{1-r^2} 6r \cdot r \, dz \, dr \, d\theta \quad : \quad 6r^2(1 - r^2 - -3) = 6r^2(4 - r^2)$$

$24r^2 - 6r^4$

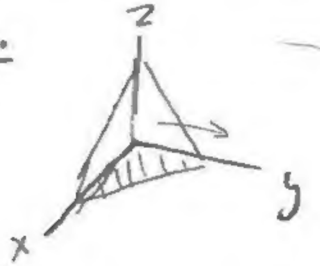
$$\int_0^{2\pi} \int_0^2 24r^2 - 6r^4 \, dr \, d\theta = 6 \int_0^{2\pi} \left(16 - \frac{32}{5}\right) d\theta$$

$$= 96 - 64 \int_0^{2\pi} d\theta$$

$$= 32(2\pi) = 64\pi \quad \blacksquare$$

Ex. Compute flux of $\vec{F} = \langle z, y, zx \rangle$ across the surface of tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Sol.

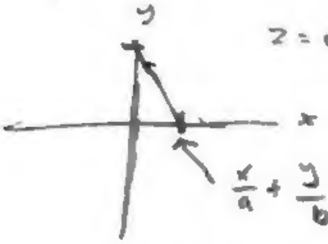


$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \vec{x} = \langle x, y, z \rangle$

$\vec{n} \cdot \vec{x} - \vec{n} \cdot \vec{p} = 0$

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$\vec{n} = \langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rangle$



$z = 0$

$R = \{(x, y, z) \mid 0 \le x \le a,\ 0 \le y \le b(1 - \frac{x}{a}),\ 0 \le z \le c(1 - \frac{x}{a} - \frac{y}{b})\}$

$z = c(1 - \frac{x}{a} - \frac{y}{b})$

$\frac{x}{a} + \frac{y}{b} = 1 \qquad y = b(1 - \frac{x}{a})$

$\frac{x}{a} = 1 \quad x = a$

$\operatorname{div}(\vec{F}) = \langle 0, +1, x \rangle = 1 + x$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx$$

$$= \int_0^a (1+x) \cdot \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= c \int_0^a (1+x) \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} = c \int_0^a (1+x) \left[b(1-\tfrac{x}{a}) \left(1 - \frac{x}{a} - \frac{b}{2b}(1 - \tfrac{x}{a})\right)\right]$$

$$= bc \tfrac{1}{2} \int_0^a (1+x)(1 - \tfrac{x}{a}) \cdot (1 - \tfrac{x}{a})\, dx$$

$$= \frac{1}{2} bc \int_0^a \left(1 + (1 - \tfrac{2}{a})x + (\tfrac{1}{a^2} - \tfrac{2}{a})x^2 + \tfrac{1}{a^2} x^3\right) dx$$

$$= \frac{1}{2} bc \left[x + \tfrac{1}{2}(1 - \tfrac{2}{a})x^2 + \tfrac{1}{3}(\tfrac{1}{a^2} - \tfrac{2}{a})x^3 + \tfrac{1}{4a^2} x^4\right]_0^a$$

$$= \frac{1}{2} bc \left(a + \tfrac{1}{2}(1 - \tfrac{2}{a})a^2 + \tfrac{1}{3}(\tfrac{1}{a^2} - \tfrac{2}{a})a^3 + \frac{a^4}{4a^2}\right)$$

$$= \frac{1}{2} abc \left(1 + \tfrac{1}{2}(a-2) + \tfrac{1}{3}(1 - 2a) + \tfrac{1}{4} a\right)$$

$$= \frac{1}{2} abc \left(\tfrac{1}{3} + a(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4})\right) = \frac{1}{6} abc \left(1 + (\tfrac{6}{4} - \tfrac{8}{4} + \tfrac{1}{4})a\right)$$

$$= \frac{1}{6} abc \left(1 + \tfrac{1}{4} a\right).$$

$\blacksquare$